Rational Homotopy Theory

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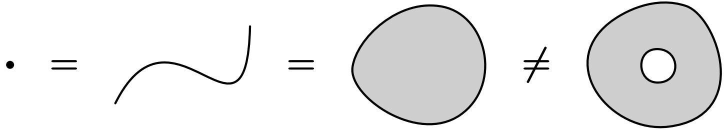
Introduction to homotopy theory

Rational homotopy theory

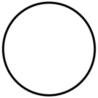
The main equivalence

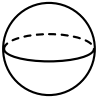
Homotopy theory

Study of spaces or shapes
with "weak equivalences"



Important spaces

$S^1 =$ 

$S^2 =$ 

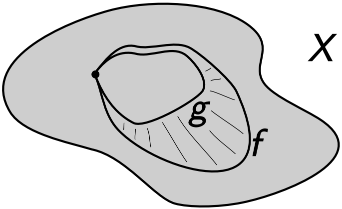
$S^3 = \cdots$

$\vdots$

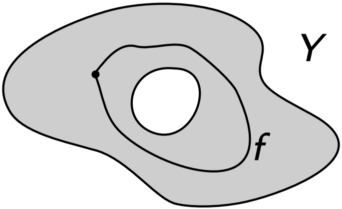
Important tool

Fundamental group:

$$\pi_1(X) = \text{maps } S^1 \to X \text{ up to homotopy}$$



$\pi_1(X) = \text{trivial}$



$\pi_1(Y) \neq \text{trivial}$

Important tools

Homotopy groups:

$$\pi_1(X) = \text{maps } S^1 \to X \text{ up to homotopy}$$

$$\pi_2(X) = \text{maps } S^2 \to X \text{ up to homotopy}$$

$$\pi_3(X) = \text{maps } S^3 \to X \text{ up to homotopy}$$

$$\vdots$$

Torsion-free

Serre proved in 1950s:

$$\text{odd } k: \quad \pi_n(S^k) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

$$\text{even } k: \quad \pi_n(S^k) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n = k, 2k-1 \\ 0 & \text{otherwise} \end{cases}$$

Introduction to homotopy theory

Rational homotopy theory

The main equivalence

Rational homotopy theory

Study of spaces
with "rational equivalences"
and "rational homotopy groups"

Rational homotopy theory

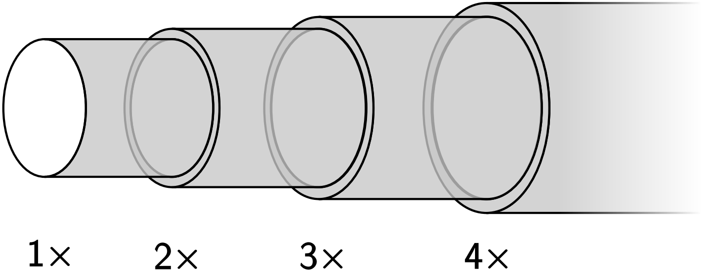
Study of spaces
with "rational equivalences"
and "rational homotopy groups"

or

Study of *rational* spaces
with weak equivalences
and ordinary homotopy groups

Rational spaces

X is *rational* if $\pi_n(X)$ is a $\mathbb{Q}$-vector space

$S^1_{\mathbb{Q}} =$ 

Main equivalence

Theorem

Homotopy theory of rational spaces

=

Homotopy theory of commutative differential graded algebras

Main equivalence (precise version)

Theorem

$$\mathbf{Ho}(\mathbf{Top}_{\mathbb{Q},1,f}) \simeq \mathbf{Ho}(\mathbf{CDGA}_{\mathbb{Q},1,f})^{op}$$

What is a cdga?

Definition

a cdga A is

- a $\mathbb{Q}$-vector space
- with a multiplication $A \otimes A \xrightarrow{\mu} A$
- with a differential $A \xrightarrow{d} A$ such that $d^2 = 0$
- with a grading $A = \bigoplus_{n \in \mathbb{N}} A^n$
- it is commutative: $xy = (-1)^{|x| \cdot |y|} yx$

Free cdga's

As always: there is a free guy: $\Lambda(\ldots)$
For example

$$\Lambda(t, dt) \text{ with } |t| = 0$$

is just polynomials in t, with its differential dt

Dictionary

rational spaces	**cdga's**
$S^n_{\mathbb{Q}}$ with n odd	$\Lambda(e)$ with $\|e\| = n$
$S^n_{\mathbb{Q}}$ with n even	$\Lambda(e, f)$ with $\|e\| = n$, $\|f\| = 2n-1$ and $df = e^2$
Eilenberg-MacLane space $K(\mathbb{Q}, n)$	$\Lambda(e)$ with $\|e\| = n$

Dictionary

rational spaces	**cdga's**
weak equivalence	weak equivalence
$\pi_n(f) : \pi_n(X) \cong \pi_n(Y)$	$H(f) : H(X) \cong H(Y)$
homotopy	homotopy
$h : X \times I \to Y$	$h : A \to B \otimes \Lambda(t, dt)$

Dictionary

rational spaces	**cdga's**
$\pi_n(X) = [S^n, X]$	$\pi^n(A) = H(Q(A))$ $\pi^n(A)^* \cong [A, \Lambda(e)]$ or $[A, \Lambda(e, f)]$
Long exact sequence of a fibration	Long exact sequence of a cofibration

Dictionary

topological n-simplex

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0 \right\}$$

cdga n-simplex

$$\Delta_n = \frac{\Lambda(x_0, \ldots x_n, dx_0, \ldots, dx_n)}{\langle \sum x_i - 1, \sum dx_i \rangle}, \qquad |x_i| = 0$$

Construction

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[d, "y"] \arrow[drr, "\Delta_{(-)}"] & & \\
\mathbf{sSet} \arrow[rr, dashed, shift left, "A"] & & {\mathbf{CDGA}_{\mathbb{Q}}}^{\mathrm{op}} \arrow[ll, dashed, shift left]
\end{tikzcd}$$

Construction

$$
\begin{tikzcd}
\boldsymbol{\Delta} \arrow[d, "y"'] \arrow[dr, "\Delta_{(-)}"] & \\
\mathbf{sSet} \arrow[r, dashed, shift left, "A"] & {\mathbf{CDGA}_{\mathbb{Q}}}^{\mathrm{op}} \arrow[l, dashed, shift left]
\end{tikzcd}
$$

$$A(X) = \mathbf{Hom}_{\mathbf{sSet}}(X, \Delta_{(-)})$$